

# Quantum Information Theory

Final exam  
Spring term 2025

From: 4th July 2025, 09h00  
To: 4th July 2025, 12h00

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## PHYS-550 – Exam – room BCH 2201

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section. **Please mark clearly which two you have answered below.** In case you don't follow this instruction (e.g. you mark all the 3 questions below, or you don't mark any of them below), then only the long Problems A and B will be graded and been considered for the total number of points.
- **Please start a new sheet for each of the long answer questions**, or a penalty of 10 points will be deduced from your total number of points.
- *Write your solutions in the indicated space. Scrap paper will not be corrected. A simple calculator (without calculator functions) is allowed.*
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

*NAME STICKER GOES HERE*

Short problems:	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
<b>Total</b>	<b>/100</b>

# Shorter (easier) questions

## 1. States

Let  $\rho^{AB}$  be a density operator on the composite Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ , and let

$$\rho^A = \text{Tr}_B[\rho^{AB}] \quad (1)$$

be its reduced state on subsystem  $A$ .

Consider a projective measurement on  $A$  given by the orthogonal projectors  $\{\Pi_i^A\}$ , with  $\sum_i \Pi_i^A = I^A$ .

- (a) Show that the probability of outcome  $i$  when measuring  $\{\Pi_i^A\}$  on the joint state  $\rho^{AB}$ ,

$$p_i = \text{Tr}\left[(\Pi_i^A \otimes I^B) \rho^{AB}\right], \quad (2)$$

can be written purely in terms of the reduced state  $\rho^A$ , namely

$$p_i = \text{Tr}_A[\Pi_i^A \rho^A]. \quad (3)$$

(4 marks)

[TJ: Answer: Many solutions accepted!]

(1)

Leverage that  $\text{Tr} = \text{Tr}_A \text{Tr}_B$  and  $\text{Tr}_B((\Pi \otimes I)\rho) = \Pi \text{Tr}_B(\rho)$  (0.5 mark deduction), then

$$\text{Tr}\left[(\Pi_i^A \otimes I^B) \rho^{AB}\right] = \text{Tr}_A\left[\text{Tr}_B\left[(\Pi_i^A \otimes I^B) \rho^{AB}\right]\right] = \text{Tr}_A\left[\Pi_i^A \text{Tr}_B(\rho^{AB})\right] = \text{Tr}_A[\Pi_i^A \rho^A].$$

(2)

Expand  $\text{Tr}$  using the measurement basis  $|\lambda_j\rangle$  for partition  $A$  (whereby  $\Pi_i^A |\lambda_j\rangle = \delta_{ij} |\lambda_j\rangle$ ) and arbitrary basis  $|k\rangle$  for partition  $B$ . Then

$$\begin{aligned} \text{Tr}\left[(\Pi_i \otimes I) \rho^{AB}\right] &= \sum_{jk} \langle \lambda_j | \langle k | (\Pi_i \otimes I) \rho^{AB} | \lambda_j \rangle | k \rangle \\ &= \sum_{jk} (\langle \lambda_j | \Pi_i \otimes \langle k | \rho^{AB} | \lambda_j \rangle | k \rangle) \\ &= \sum_k \langle \lambda_i | \langle k | \rho^{AB} | \lambda_i \rangle | k \rangle \\ &= \langle \lambda_i | \sum_k [(I \otimes \langle k | \rho^{AB} (I \otimes | k \rangle)] | \lambda_i \rangle \quad (\text{nani?!}) \\ &= \langle \lambda_i | \text{Tr}_B(\rho^{AB}) | \lambda_i \rangle \\ &= \text{Tr}[\Pi_i \rho^A] \end{aligned}$$

(3)

Go backwards. Beginning from

$$\rho_A = \text{Tr}_B(\rho^{AB}) = \sum_k (I \otimes \langle k|) \rho^{AB} (I \otimes |k\rangle)$$

Therefore

$$\begin{aligned} \text{Tr}[\Pi_i \rho^A] &= \text{Tr} \left[ \Pi_i \sum_k (I \otimes \langle k|) \rho^{AB} (I \otimes |k\rangle) \right] \\ &= \sum_k \text{Tr} [\Pi_i (I \otimes \langle k|) \rho^{AB} (I \otimes |k\rangle)] && \text{(linearity)} \\ &= \sum_k \text{Tr} [(I \otimes |k\rangle) \Pi_i (I \otimes \langle k|) \rho^{AB}] && \text{(cyclicity)} \\ &= \sum_k \text{Tr} [(\Pi_i \otimes |k\rangle \langle k|) \rho^{AB}] && \text{(magic?!)} \\ &= \text{Tr} \left[ \left( \Pi_i \otimes \left( \sum_k |k\rangle \langle k| \right) \right) \rho^{AB} \right] && \text{(linearity)} \\ &= \text{Tr} [(\Pi_i \otimes I) \rho^{AB}] \end{aligned}$$

(4)

Element-wise! Express  $\rho^{AB}$  in a basis where the first partition is the basis of measurement projectors,  $\{|\lambda_j\rangle\}$ . Substitute

$$\rho^{AB} = \sum_{abcd} \rho_{abcd} |\lambda_a\rangle \langle \lambda_b| \otimes |c\rangle \langle d|$$

into

$$\begin{aligned} \text{Tr} [(\Pi_i \otimes I) \rho^{AB}] &= \text{Tr} \left[ (\Pi_i \otimes I) \sum_{abcd} \rho_{abcd} |\lambda_a\rangle \langle \lambda_b| \otimes |c\rangle \langle d| \right] \\ &= \text{Tr} \left[ \sum_{abcd} \rho_{abcd} \Pi_i |\lambda_a\rangle \langle \lambda_b| \otimes |c\rangle \langle d| \right] \\ &= \text{Tr} \left[ \sum_{bcd} \rho_{ibcd} |\lambda_i\rangle \langle \lambda_b| \otimes |c\rangle \langle d| \right] \\ &= \sum_{bcd} \rho_{ibcd} \text{Tr} [|\lambda_i\rangle \langle \lambda_b|] \text{Tr} [|c\rangle \langle d|] && \text{(linearity)} \\ &= \sum_c \rho_{iicc} \end{aligned}$$

Meanwhile, evaluating the partial trace with respect to the basis of the second

partition:

$$\begin{aligned}
 \text{Tr}[\Pi_i \rho^A] &= \text{Tr}[\Pi_i \text{Tr}_B(\rho^{AB})] = \text{Tr} \left[ \Pi_i \sum_k (I \otimes \langle k|) \rho^{AB} (I \otimes |k\rangle) \right] \\
 &= \text{Tr} \left[ \Pi_i \sum_k (I \otimes \langle k|) \sum_{abcd} \rho_{abcd} |\lambda_a\rangle \langle \lambda_b| \otimes |c\rangle \langle d| (I \otimes |k\rangle) \right] \\
 &= \text{Tr} \left[ \Pi_i \sum_k \sum_{abcd} \rho_{abcd} |\lambda_a\rangle \langle \lambda_b| \langle k|c\rangle \langle d|k\rangle \right] \\
 &= \text{Tr} \left[ \Pi_i \sum_k \sum_{ab} \rho_{abkk} |\lambda_a\rangle \langle \lambda_b| \right] \\
 &= \text{Tr} \left[ \sum_k \sum_b \rho_{ibkk} |\lambda_i\rangle \langle \lambda_b| \right] \\
 &= \sum_{kb} \rho_{ibkk} \text{Tr}[|\lambda_i\rangle \langle \lambda_b|] \quad (\text{linearity}) \\
 &= \sum_k \rho_{iikk}
 \end{aligned}$$

which is an equivalent sum to our previous term.

]

Penalties:

- (-0.5 mark) Leveraging  $\text{Tr}_B((\Pi \otimes I)\rho) = \Pi \text{Tr}_B(\rho)$
- (-1 mark) Incorrect dimensions in algebra
- (-1.5 mark) Assumed  $\rho^{AB}$  was pure.
- (-1.5 mark) Assumed  $\rho^{AB}$  was separable.

(b) Explain what is meant by ‘purifying’ a mixed state  $\rho$ .

(2 marks)

[TJ: Answer:

- (1 mark) Find a **pure** state in a **larger** Hilbert space.
- (1 mark) Original mixed state is a **reduced state** of (or otherwise found via the **partial trace** from) the pure state.

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Penalties:

- (-0.5 mark) Suggesting purification was a physical process (e.g. whereby  $\rho$  “interacts” with an ancilla)

(c) Show that it is possible to purify any state  $\rho$ . How large a purifying system is required to do so?

(4 marks)

[TJ: Answer: Since a density matrix is positive semidefinite (**bonus 0.5 marks**), we can always (without loss of generality) express  $\rho$  in a basis  $\{|e_j\rangle\}$  with positive coefficients  $p_j$  via the Schmidt decomposition.

$$\rho \equiv \sum_j p_j |e_j\rangle \langle e_j| \quad (1 \text{ mark})$$

Consider now the below pure state in a larger Hilbert space

$$|\psi\rangle = \sum_j \sqrt{p_j} |e_j\rangle |j\rangle \quad (1 \text{ mark})$$

where  $\{|j\rangle\}$  are orthonormal states of an ancillary system.

We can show  $|\psi\rangle$  is a purification of  $\rho$ , i.e. that

$$\rho = \text{Tr}_B(|\psi\rangle \langle \psi|) \quad (0.5 \text{ marks})$$

This is straightforward to show (for **0.5 marks**):

$$\begin{aligned} |\psi\rangle \langle \psi| &= \left( \sum_j \sqrt{p_j} |e_j\rangle |j\rangle \right) \left( \sum_k \sqrt{p_k} \langle e_k| \langle k| \right) \\ &= \sum_{jk} \sqrt{p_j p_k} |e_k\rangle \langle e_j| \otimes |j\rangle \langle k| \end{aligned}$$

$$\begin{aligned} \text{Tr}_B(|\psi\rangle \langle \psi|) &= \sum_l (I \otimes \langle l|) |\psi\rangle \langle \psi| (I \otimes |l\rangle) \\ &= \sum_{ljk} \sqrt{p_j p_k} |e_k\rangle \langle e_j| \langle j|l\rangle \langle k|l\rangle \\ &= \sum_l \sqrt{p_l p_l} |e_l\rangle \langle e_l| = \rho \quad (|p_l| = p_l \text{ since } p_l > 0) \end{aligned}$$

The dimension of the ancillary system need not be any larger than that of  $\rho$  (**1 mark**), and is lower-bounded by the Schmidt rank of  $\rho$  (**bonus 1 mark**). ]

Penalties:

- (-1 mark) Assuming a specific Schmidt rank (usually 2, i.e. “let  $\rho = p_1\sigma_1 + (1 - p_1)\sigma_2$ ”).

## 2. Measurements

Consider the following three-outcome POVM on a qubit:

$$E_i = \frac{1}{3}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma}), \quad i = 1, 2, 3, \quad (4)$$

where  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  are the Pauli matrices.

- (a) What constraints do we need on the  $\mathbf{n}_i$  vectors to ensure that this is a valid POVM?

(6 marks)

Answer:  $E_i$ 's need to be positive and  $\sum_i E_i = I$ . To ensure the positivity, we compute the eigenvalues of  $E_i$  and find that  $\text{eig}(E_i) = \frac{1}{3}(1 \pm \sqrt{n_{i,x}^2 + n_{i,y}^2 + n_{i,z}^2}) = \frac{1}{3}(1 \pm |\mathbf{n}_i|)$ . So the positivity requires that  $|\mathbf{n}_i| \leq 1 \forall i$ . (A short cut here would be to recognise that the expression for the POVM elements is proportional to that for a state and so you can spot immediately that we need  $|\mathbf{n}_i| \leq 1 \forall i$ ).

In addition,  $\sum_i E_i = I + \frac{1}{3}(\sum_i \mathbf{n}_i \cdot \boldsymbol{\sigma}) = I$ . Hence, this requires  $\sum_i \mathbf{n}_i = 0$ .

3 pts for positivity and 3 pts for the sum of identity. Most get 6 pts. A few ppl ignore the positivity requirement and hence get 3 pts.

- (b) Suppose we have a state  $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})$  where  $\|\mathbf{r}\| \leq 1$ . Show that the probability of measuring outcome  $i$  is given by

$$p_i = \text{Tr}(E_i \rho) = \frac{1}{3}(1 + \mathbf{n}_i \cdot \mathbf{r}). \quad (5)$$

(2 marks) Answer:

$$p_i = \text{Tr}[\rho E_i] = \text{Tr}[\rho \cdot \frac{1}{3}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma})] \quad (6)$$

$$= \text{Tr}[\frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}) \cdot \frac{1}{3}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma})] \quad (7)$$

$$= \text{Tr}[\frac{1}{6}I + \frac{1}{6}\mathbf{r} \cdot \boldsymbol{\sigma} + \frac{1}{6}\mathbf{n}_i \cdot \boldsymbol{\sigma} + \frac{1}{6}(\mathbf{r} \cdot \boldsymbol{\sigma})(\mathbf{n}_i \cdot \boldsymbol{\sigma})] \quad (8)$$

$$= \frac{1}{3}(1 + \mathbf{n}_i \cdot \mathbf{r}). \quad (9)$$

2 pts for the proof. Almost everyone get all 2 pts.

- (c) Specify a choice in the vectors  $\mathbf{n}_i$  such that each POVM element is proportional to a projector.

Describe what measurement procedure this case corresponds to.

(2 marks) Answer: We are looking for  $n_i$ 's, such that they fulfil the requirements in a) and also  $E_i^2 = \alpha_i E_i \forall i$  where  $\alpha_i > 0$ . Solving this equation,

$$(\frac{1}{3}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma}))^2 = \alpha_i \frac{1}{3}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma}) \quad (10)$$

$$\frac{1}{9}(I + |\mathbf{n}_i|^2 \cdot I + 2\mathbf{n}_i \cdot \boldsymbol{\sigma}) = \frac{\alpha_i}{3}I + \frac{\alpha_i}{3}\mathbf{n}_i \cdot \boldsymbol{\sigma} \quad (11)$$

$$\frac{1}{9}(1 + |\mathbf{n}_i|^2)I + \frac{2}{9}\mathbf{n}_i \cdot \boldsymbol{\sigma} = \frac{\alpha_i}{3}I + \frac{\alpha_i}{3}\mathbf{n}_i \cdot \boldsymbol{\sigma}, \quad (12)$$

we find  $\alpha_i = 2/3$  and  $|\mathbf{n}_i| = 1$ . Together with the requirement  $\sum_i \mathbf{n}_i = 0$ , we could choose, for example, an equilateral triangle in x-y plane of the Bloch space,

$n_1 = (1, 0, 0), n_2 = (-1/2, \sqrt{3}/2, 0)$  and  $n_3 = (-1/2, -\sqrt{3}/2, 0)$ .

1 pt if they ignored “proportional to” and find measurements contradicting to the requirement in a). But 2 pts if they (mis)interpreted the (unintentionally vague) question as finding a projective measurement argued that it’s impossible. 0 pt is they proposed something that is not, or not proportional to projectors.

### 3. Shot noise

Consider the observable  $O = a\sigma_x + b\sigma_z$  and a generic initial state  $\rho = |0\rangle\langle 0|$ .

Suppose we measure this observable by first rotating  $\rho$  into the eigenbasis of  $O$ .

(a) Bound the convergence of this estimator using Chebyshev’s inequality.

(5 marks) Answer: Chebyshev’s inequality for shot noise is

$$\Pr(|\hat{O} - \text{Tr}[O\rho]| \geq \epsilon) \leq \frac{\text{Var}_\rho(O)}{N\epsilon^2}$$

where

$$\text{Var}_\rho(O) = \text{Tr}[O^2\rho] - \text{Tr}[O\rho]^2 \quad (13)$$

$$= \text{Tr}[13I\rho] - \text{Tr}[|0\rangle\langle 0|(3X + 2Z)]^2 \quad (14)$$

$$= 13 - 2^2 = 9. \quad (15)$$

Consequently,  $N \geq \frac{\text{Var}_\rho(O)}{\delta\epsilon^2} = \frac{9}{\delta\epsilon^2}$ .

3 points for the method of using Chebyshev to compute the variance, and 2 point for calculation and finding the bound.

(b) Bound the convergence of this estimator using Hoeffding’s inequality.

(5 marks) Answer: Hoeffding’s inequality

$$\Pr(|\hat{O} - \text{Tr}[O\rho]|) \leq 2 \exp\left(-\frac{2N\epsilon^2}{\Delta\lambda^2}\right)$$

where  $\Delta\lambda = \lambda_{max} - \lambda_{min}$ . And here we find that the eigenvalues of  $O$  are  $\pm\sqrt{13}$  and hence  $\Delta\lambda^2 = 52$ . Thus, to obtain a probability of at least  $1 - \delta$ , we need

$$N \geq -\frac{\Delta\lambda^2}{2\epsilon^2} \log\left(\frac{\delta}{2}\right) = -\frac{26}{\epsilon^2} \log\left(\frac{\delta}{2}\right).$$

2 points for Hoeffding’s inequality, 2 point for understanding its terms correctly, and 1 points for calculation.

### 4. Entropies

Show that a pure state  $|\psi\rangle_{AB}$  is entangled between subsystems  $A$  and  $B$  if and only if the conditional entropy is less than zero i.e.,  $S(A|B) < 0$ .

(10 marks) For a pure state  $|\psi\rangle_{AB}$ , we have  $S(AB) = 0$ . Therefore,  $S(A|B) = S(AB) - S(B) = -S(B)$  so we just need to show that  $|\psi\rangle_{AB}$  is entangled iff  $S(B) > 0$  (5 points). To prove it, we have that  $|\psi\rangle_{AB}$  is entangled iff the reduced state  $\rho_B$  is mixed iff  $S(\rho_B) > 0$  (5 points for the proof). Half of the proof (if instead of iff) gives up to 3 points (instead of 5). Proving by contrapositive also work (using  $S(B) \geq 0$  with the equality iff  $\rho_B$  is pure iff  $|\psi\rangle$  is separable). In the case where something is missing,  $|\psi\rangle_{AB}$  entangled iff  $\rho_B$  is mixed (2 points) and  $\rho_B$  mixed iff  $S(\rho_B) > 0$  (3 points).

## 5. Entanglement

Consider the two-qubit states  $\rho_p$  defined by

$$\rho_p = p |\Psi^-\rangle\langle\Psi^-| + (1-p) \frac{I}{4}, \quad (16)$$

where  $I$  is the  $4 \times 4$  identity,  $p \in (0, 1)$ , and

$$|\Psi^-\rangle = \frac{|0, 1\rangle - |1, 0\rangle}{\sqrt{2}} \quad (17)$$

is the singlet (maximally entangled) state.

Determine the range of  $p$  for which  $\rho_p$  is separable and for which it is entangled.

(10 marks)

Mentioning *PPT* (or Peres-Horodecki criterion) gives 2 points. Correctly apply partial transpose gives 2 points. Recognising the two first trivial eigenvalues (i.e.  $(1+p)/4$ ) 1 point. Finding the other two which are also  $(1+p)/4$  and more importantly  $(1-3p)/4$  gives 2 points (i.e. 3 points for the spectrum). This implies  $\rho$  separable if  $p \geq -1$  and  $p \leq 1/3$  (1 point). Then, finally 2 points for the intersection with  $p \in (0, 1)$  such that separable iff  $p \in (0, 1/3]$  and/or entangled iff  $p \in (1/3, 1)$  (i.e. 3 points for the conclusion from the spectrum).

-Bonus point for checking the values for which  $\rho$  is a state (checking the eigenvalues are positive for  $p \in (0, 1)$ ) but no mistake allowed to get the bonus.

-If the eigenvalues are wrong, but the result is consistent might still give 2 points for the conclusion and if they only mention PPT criterion but clearly describe the protocol, they can get additional 2 points.

-Alternatively, after mentioning Peres-Horodecki criterion (2 points), they can use the *SWAP* as entanglement witness i.e. using  $\text{Tr}[SWAP\rho] < 0$  iff entangled. They get 3 points for *SWAP* $\rho$  and 2 points for the trace (i.e. up to 5 points for the correct trace), then 3 points for the conclusion with the trace.

(Still up to 1 points using the entropy of the reduced state which is wrong since we have a mixed state here and not a pure state, so any single mistake is not allowed if they did that.)

# Longer (harder) questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

## Question A - Fidelity

The goal of this problem is to derive, and use, the following alternative expression for the fidelity:

$$F(P, Q) := \text{Tr} \left[ \sqrt{P^{\frac{1}{2}} Q P^{\frac{1}{2}}} \right] = \text{Tr} \left[ \sqrt{PQ} \right], \quad (18)$$

which holds if  $P$  is positive definite.

Let matrices  $A, B > 0$ , we call

$$A \# B := A^{\frac{1}{2}} \sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} A^{\frac{1}{2}} \quad (19)$$

the geometric mean between  $A$  and  $B$ .

- (a) Show that the geometric mean  $X = A \# B$  is the unique positive solution of the Riccati equation  $XA^{-1}X = B$ .

*Hint: You can first show that the geometric mean solves this equation, and then show the uniqueness.*

**Answer:**

- a) By direct substitution  $A \# B := A^{\frac{1}{2}} \sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} A^{\frac{1}{2}}$ , we find

$$A^{\frac{1}{2}} \sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} A^{\frac{1}{2}} A^{-1} A^{\frac{1}{2}} \sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} A^{\frac{1}{2}} = B. \quad (20)$$

**+3 points**

Hence,  $A \# B$  is a solution of  $XA^{-1}X = B$ . We next show that this solution is unique.

Suppose  $X, Y > 0$  and both solve the Riccati equation, i.e.  $XA^{-1}X = B = YA^{-1}Y$ .

Then we have

$$(A^{-\frac{1}{2}} X A^{-\frac{1}{2}})^2 = A^{-\frac{1}{2}} X A^{-1} X A^{-\frac{1}{2}} = A^{-\frac{1}{2}} Y A^{-1} Y A^{-\frac{1}{2}} = (A^{-\frac{1}{2}} Y A^{-\frac{1}{2}})^2, \quad (21)$$

and hence, by uniqueness of positive definite square roots,  $A^{-\frac{1}{2}} X A^{-\frac{1}{2}} = A^{-\frac{1}{2}} Y A^{-\frac{1}{2}}$ .

Since  $A^{-\frac{1}{2}}$  is invertible, we conclude that  $X = Y$ .

**+6 points, can be also proved via e.g. positivity of operators.**

(9 marks)

(b) Next, let  $P, Q$  be two positive definite density matrices. Show that

$$\text{Tr} \left[ \sqrt{PQ} \right] = \text{Tr} \left[ \sqrt{P^{\frac{1}{2}}QP^{\frac{1}{2}}} \right]. \quad (22)$$

*Hint: Ricatti equation will be helpful if you find a right way to use it.*

(9 marks)

b) Thus we have  $PQ = P(P^{-1}\#Q)P(P^{-1}\#Q)$ . And then

$$\sqrt{PQ} = PP^{-1}\#Q \quad (23)$$

$$= P \left( P^{-\frac{1}{2}} \sqrt{P^{\frac{1}{2}}QP^{\frac{1}{2}}} P^{-\frac{1}{2}} \right) \quad (24)$$

$$= P^{\frac{1}{2}} \sqrt{P^{\frac{1}{2}}QP^{\frac{1}{2}}} P^{-\frac{1}{2}}. \quad (25)$$

Using the cyclic property of trace, we obtain

$$\text{Tr} \left[ \sqrt{PQ} \right] = \text{Tr} \left[ \sqrt{P^{\frac{1}{2}}QP^{\frac{1}{2}}} \right] = F(P, Q). \quad (26)$$

4 points for mastering the matrix calculations here, and 5 points for correctly using Ricatti.

Penalty: - 1 or 2 points if in the proof one don't clearly argue the steps and clearly explain which identity is used.

(c) Hence, or otherwise, compute the fidelity between the following states:

i)  $\rho = \frac{1}{2}(I + \sigma_x)$  and  $\sigma = \frac{1}{2}(I + \sigma_y)$

ii)  $\rho = \frac{1}{2}(I + 0.5\sigma_x)$  and  $\sigma = \frac{1}{2}(I + 0.8\sigma_x)$

iii)  $\rho = \frac{1}{2}(|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|)$  and  $\sigma = \frac{1}{2}(|\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|)$

(7 marks)

i). (2 points)  $\rho\sigma = \frac{1}{4}(I + \sigma_X + \sigma_Y + i\sigma_Z) \Rightarrow \sqrt{\rho\sigma} = \frac{\sqrt{2}}{4}(I + \sigma_X + \sigma_Y + i\sigma_Z) \Rightarrow F = \frac{1}{\sqrt{2}}$ .  
Alternatively, recognise that you are looking at the pure state fidelity (i.e., inner product) between the positive eigenstates of  $\sigma_x$  and  $\sigma_y$  and see immediately that this is  $\frac{1}{\sqrt{2}}$ .  
(Don't drop a mark for a correct calculation that used the squared version of the Fidelity. Do drop a mark if the calculation was using the non-squared version and then a square magically appeared to give  $1/2$ ).

ii). (3 points)  $\rho\sigma = \frac{1}{4}(1.4I + 1.3\sigma_X) \Rightarrow \sqrt{\rho\sigma} = \frac{\sqrt{10+3\sqrt{30}}}{40}I + \frac{-\sqrt{10+3\sqrt{30}}}{40}\sigma_X \Rightarrow F = \frac{\sqrt{10+3\sqrt{30}}}{20}$ .

iii). (2 points)  $F = 0$  since  $\rho\sigma = 0$ .

Alternatively, notice that we are looking at the mixed state fidelity between two states that are diagonal in the same basis and see that they have non overlapping support and so the fidelity is 0.

## Question B - Majorization and entanglement

Let  $\mathbf{x} = [x_1, x_2, \dots, x_d] \in \mathbb{R}^d$  and  $\mathbf{y} = [y_1, y_2, \dots, y_d] \in \mathbb{R}^d$  be two vectors such that  $x_1 \geq x_2 \geq \dots \geq x_d$  and  $y_1 \geq y_2 \geq \dots \geq y_d$ . We say that  $\mathbf{x}$  is *majorized* by  $\mathbf{y}$  if

$$\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i \quad \forall j \in \{1, 2, \dots, d-1\}, \quad (27)$$

$$\sum_{i=1}^d x_i = \sum_{i=1}^d y_i. \quad (28)$$

Equivalently,  $\mathbf{x}$  is majorized by  $\mathbf{y}$  if and only if there exists a doubly stochastic matrix  $D$  such that  $D\mathbf{y} = \mathbf{x}$ . (A doubly stochastic matrix is one such that  $\sum_i D_{ij} = 1$  for all  $j$  and  $\sum_j D_{ij} = 1$  for all  $i$ .)

Consider two  $d$ -dimensional density matrices  $\rho, \sigma \in \mathbb{C}^d$  such that

$$\rho = \sum_{i=1}^n p_i U_i \sigma U_i^\dagger, \quad (29)$$

where  $U_1, U_2, \dots, U_n$  are unitary and  $\sum_{i=1}^n p_i = 1$ .

- (a) Show that the eigenvalues of  $\rho$  are majorized by the eigenvalues of  $\sigma$ .  
(6 marks)

First, consider the eigendecompositions of  $\rho$  and  $\sigma$ :

$$\rho = \sum_{i=1}^d \lambda_i(\rho) |\psi_i\rangle \langle \psi_i|$$

and

$$\sigma = \sum_{i=1}^d \lambda_i(\sigma) |\phi_i\rangle \langle \phi_i|,$$

where  $\lambda_i(\rho)$  and  $\lambda_i(\sigma)$  are the  $i$ -th eigenvalues (listed in decreasing order) of  $\rho$  and  $\sigma$  respectively, and both  $|\psi_i\rangle$  and  $|\phi_i\rangle$  are the corresponding eigenvectors (forming two orthonormal basis). (1 point)

Then, we can express  $\rho$  as function of the eigenvalues of  $\sigma$  as follows

$$\begin{aligned} \rho &= \sum_{l=1}^d \lambda_l(\sigma) \sum_{i=1}^n p_i U_i |\phi_l\rangle \langle \phi_l| U_i^\dagger \quad (1 \text{ point}) \\ &= \sum_{k=1}^d \lambda_k(\rho) |\psi_k\rangle \langle \psi_k|. \end{aligned}$$

Now, we can express  $\lambda_k(\rho)$  as function of the eigenvalues of  $\sigma$  as follows

$$\begin{aligned}\lambda_k(\rho) &= \langle \psi_k | \rho | \psi_k \rangle \\ &= \sum_{l=1}^d \lambda_l(\sigma) \underbrace{\sum_{i=1}^n p_i \langle \psi_k | U_i | \phi_l \rangle \langle \phi_l | U_i^\dagger | \psi_k \rangle}_{\equiv D_{k,l}}. \quad (1 \text{ point})\end{aligned}$$

Now, we can see that the matrix  $D$  with coefficients defined above is a doubly stochastic matrix since all the coefficients are positive and  $\sum_k D_{k,l} = \sum_l D_{k,l} = 1$  (the proof is quite straightforward). Finally, we have  $\lambda(\rho) = D\lambda(\sigma)$  which implies  $\lambda(\rho)$  is majorized by  $\lambda(\sigma)$ . (2 points for showing  $D$  is doubly stochastic and 1 point for the conclusion) - Alternatively, we could use the projector onto the first  $J$  largest eigenvalues of  $\rho$  i.e.  $P_J = \sum_{k=1}^J |\psi_k\rangle \langle \psi_k|$  as follows:

$$\begin{aligned}\text{Tr}[\rho P_J] &= \sum_{k=1}^J \lambda_k(\rho) \\ &= \sum_{i=1}^n p_i \text{Tr}[P_J U_i \sigma U_i^\dagger] \\ &\leq \sum_{i=1}^n p_i \sum_{k=1}^J \lambda_k(\sigma) \\ &= \sum_{k=1}^J \lambda_k(\sigma),\end{aligned}$$

where the inequality is obtained by taking the maximum over projectors of rank  $J$  such that  $\max_{P: P^2=P, \text{rank}(P)=J} [\text{Tr}[P\sigma]] = \sum_{k=1}^J \lambda_k(\sigma)$ . (6 points)

-Another alternative to get some points which is more intuitive but less formal is to realise that the spectrum of  $\sigma$  and  $U_i \sigma U_i^\dagger$  coincides (2 points). So the spectrum of  $\rho$  is a convex combination of permutations of the spectrum of  $\sigma$  (2 points). A convex combination of permutation can be represented by a doubly stochastic matrix so we can prove the claim (2 points).

- (b) Suppose that  $B, C$  are  $n$  dimensional complex matrices such that  $B^\dagger B = C^\dagger C$ . Then show that there must exist a unitary  $U \in \mathbb{C}^{n \times n}$  such that  $B = UC$ .

(4 marks) We need to consider the singular value decomposition (or alternatively polar decomposition) of  $B$  and  $C$  which is unique (2 points for using/mentioning SVD or polar decomposition).

$$B = V_B \Sigma_B W_B^\dagger$$

and

$$C = V_C \Sigma_C W_C^\dagger.$$

Since,  $B^\dagger B = C^\dagger C$  we have  $W_C = W_B = W$  and  $\Sigma_C = \Sigma_B = \Sigma$  so that  $B = V_B \Sigma W^\dagger$  and  $C = V_C \Sigma W^\dagger$  which leads us to  $B = V_B V_C^\dagger C$  and since both  $V_B$  and  $V_C$  are unitaries, we have  $U = V_B V_C^\dagger$  is unitary (2 points for the proof). The proof with polar decomposition is very similar.

Attempts using further assumptions on  $B$  and/or  $C$  cannot get more than 2 points. Assuming both ranks are equal and using the fact that there exists a matrix  $A$  such that  $B = AC$  and rigorously showing that  $A$  is unitary can get 2 points. Using the fact that  $B$  and/or  $C$  are invertible get 1 point. Showing only one side of the relation (if  $B = UC$ , then  $B^\dagger B = C^\dagger C$ ) using  $B^\dagger B = B^\dagger U^\dagger U B$  to get 1 point.

Consider now a bipartite pure state  $|\psi^{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$  between Alice and Bob. Alice and Bob perform the following LOCC operation:

- i. Alice performs a POVM with Kraus operators  $M_1, M_2, \dots, M_n$ .
- ii. Alice sends the measurement outcome to Bob.
- iii. Depending on the measurement outcome  $k \in \{1, 2, \dots, n\}$ , Bob applies a channel  $\mathcal{E}_k$  on his state.

Suppose that the measurement operators  $M_k$  and channels  $\mathcal{E}_k$  are chosen such that after this protocol Alice and Bob still share a pure state  $|\phi^{AB}\rangle$ .

The output of this protocol is thus:

$$|\phi\rangle\langle\phi| = \sum_i (I \otimes \mathcal{E}_i) ((I \otimes M_i) |\psi^{AB}\rangle\langle\psi^{AB}| (I \otimes M_i^\dagger)) \quad (30)$$

- (c) If  $\rho_\psi = \text{Tr}_B(|\psi^{AB}\rangle\langle\psi^{AB}|)$  and  $\rho_\phi = \text{Tr}_B(|\phi^{AB}\rangle\langle\phi^{AB}|)$ , show that  $\forall i \in \{1, 2, \dots, k\}$ ,

$$M_i \rho_\psi M_i^\dagger \propto \rho_\phi. \quad (31)$$

*Hint- given that the output state is **pure**, what does this imply about each of the terms  $(I \otimes \mathcal{E}_i) ((I \otimes M_i) |\psi^{AB}\rangle\langle\psi^{AB}| (I \otimes M_i^\dagger))$  ?*

(7 marks) Use the hint and explain that each term in the sum should be proportional to  $|\phi\rangle\langle\phi|$  (pure state written as convex combination of states implies that each term is proportional to the pure state since a pure state cannot be written as convex combination of different states) gives up to 3 points (if it's clearly explained).

(Approaches that incorrectly argued that the output being pure implied that the input state was a product got maximum 1 mark.)

Then, we have that

$$\text{Tr}_B[(I \otimes \mathcal{E}_i) ((I \otimes M_i) |\psi^{AB}\rangle\langle\psi^{AB}| (I \otimes M_i^\dagger))] = \text{Tr}_B[ ((I \otimes M_i) |\psi^{AB}\rangle\langle\psi^{AB}| (I \otimes M_i^\dagger)) ],$$

since  $\mathcal{E}_i$  is trace preserving (2 points) and

$$\text{Tr}_B[(I \otimes \mathcal{E}_i)((I \otimes M_i)|\psi^{AB}\rangle\langle\psi^{AB}|(I \otimes M_i^\dagger))] = M_i\rho_\psi M_i^\dagger,$$

which means  $p_i\rho_\phi = M_i\rho_\psi M_i^\dagger$ . (2 points).

The final 4 points can also be obtained by showing the equation above with another legitimate approach (e.g. via Schmidt decomposition).

(Approaches that made additional assumptions on the inputs - e.g. that its diagonal or product - get at most two marks.

- (d) Show now that the Schmidt coefficients of  $|\psi^{AB}\rangle$  are majorized by those of  $|\phi^{AB}\rangle$ .

*Hint - use (b) and (c) to express  $\rho_\psi$  as a convex combination of unitaries drawn from a probability distribution and applied on  $\rho_\phi$ .*

(6 marks)

Here, we use (b) with  $p_i\rho_\phi = M_i\rho_\psi M_i^\dagger$  (1 point), where we identify  $C = \sqrt{\rho_\psi}M_i^\dagger$  and  $B = \sqrt{p_i}\sqrt{\rho_\phi}$  so we have  $U_i\sqrt{p_i}\sqrt{\rho_\phi} = \sqrt{\rho_\psi}M_i^\dagger$  for some unitary  $U_i$  (1 point). So, we have  $\sqrt{\rho_\psi}M_i^\dagger M_i\sqrt{\rho_\phi} = p_i U_i\rho_\phi U_i^\dagger$  (1 point). By summing over  $i$  both side (1 point) and using Kraus condition for the  $M_i$  (1 point), we are left with  $\rho_\psi = \sum_i p_i U_i\rho_\phi U_i^\dagger$  and we can use (a) to conclude (1 point). Bonus point for realizing that the eigenvalues of  $\rho_\psi$  and  $\rho_\phi$  correspond to the square amplitude of the Schmidt coefficients of the states  $|\psi\rangle$  and  $|\phi\rangle$  respectively so it's not majorization but weak majorization.

Alternatively, saying that we are looking for a form  $\rho_\psi = \sum_i p_i U_i\rho_\phi U_i^\dagger$  to use (a) gives 2 points (without any proof) and using  $p_i\rho_\phi = M_i\rho_\psi M_i^\dagger$  gives 1 point additionally.

Mentioning Nielsen here is 2 points to be nice but should have been clear based on the points and next question that we wanted an explicit calculation here rather than an appeal to this theorem.

- (e) What major theorem in entanglement theory is your answer to (d) an example of? It is consistent or inconsistent with that theorem? Explain your reasoning.

(2 marks)

Mention Nielsen's majorization theorem and correct statement that it is consistent was enough to get 2 points here (only 1 point if it's not well explained but mentioned or partially correct).

## Question C - Vectorization

Quantum information theorists love computing the average expectation values over random unitaries.

To do so, we often make use of the following standard identities:

$$\int_{U(d)} U \otimes U^* dU = \frac{1}{d} |\text{vec}(I_d)\rangle\langle\text{vec}(I_d)| \quad (32)$$

and

$$\begin{aligned} & \int_{U(d)} U^{\otimes 2} \otimes (U^*)^{\otimes 2} dU \\ &= \frac{1}{d^2 - 1} \left( |\text{vec}(I_{d^2})\rangle\langle\text{vec}(I_{d^2})| - \frac{1}{d} |\text{vec}(I_{d^2})\rangle\langle\text{vec}(F)| - \frac{1}{d} |\text{vec}(F)\rangle\langle\text{vec}(I_{d^2})| + |\text{vec}(F)\rangle\langle\text{vec}(F)| \right) \end{aligned} \quad (33)$$

where  $d$  is the dimension of  $U$ ,  $U(d)$  is the ensemble of all unitaries of dimension  $d$ ,  $I_d$  is the identity of dimension  $d$ , and  $F := \sum_{i,j=1}^d |i,j\rangle\langle j,i|$  is the SWAP operator.

*Remark: You do not need to understand where these identities come from, or even fully understand them, to answer this question! (But if you've previously studied group averaging maybe you can see how they would be derived.)*

This question will walk you through how to use vectorization to use these identities.

Let us start by supposing that given some initial state  $\rho$  and observable  $O$  we are interested in computing the expectation value

$$\int_{U(d)} \text{Tr}[U\rho U^\dagger O] dU. \quad (34)$$

(a) Show that

$$\mathcal{A} := \text{Tr}[U\rho U^\dagger O] = \langle\text{vec}(O)|U \otimes U^*|\text{vec}(\rho)\rangle \quad (35)$$

(1 marks)

Answer: Using the fact that trace of two operators equals the inner product of their vectorization and the Ricochet property, we obtain the statement. [1pt for the proof.](#) They are expected to say that they use the vectorization identity. But proof steps that intrinsically imply that they used the identity will also be accepted.

(b) Hence use Eq. (32) to compute  $\int_{U(d)} \text{Tr}[U\rho U^\dagger O] dU$ . What is its value if  $O$  is a Pauli matrix?

(3 marks)

Answer: Combine eq. 5 and 8, we find  $\int dU \mathcal{A} = \frac{1}{d} \langle \text{vec}(O) | \text{vec}(I_d) \rangle \langle \text{vec}(I_d) | \text{vec}(\rho) \rangle = \frac{1}{d} \text{Tr}[O] \text{Tr}[\rho] = \frac{1}{d} \text{Tr}[O]$  which equals 0 if  $O$  is a Pauli matrix. 2 points for the expression. Both in terms of traces or inner product of vectorizations will be accepted, as we didn't ask them to write the expressions as traces. 1 pt for finding the value for  $O$  being Pauli observables.

- (c) Now show that  $\text{Tr}[U\rho U^\dagger O]^2 = \langle \text{vec}(O^{\otimes 2}) | U^{\otimes 2} \otimes U^{*\otimes 2} | \text{vec}(\rho^{\otimes 2}) \rangle$ .  
(2 marks)

Answer: As trace of tensor square of operators equals the square of their trace, we have  $\text{Tr}[U\rho U^\dagger O]^2 = \text{Tr}[U^{\otimes 2} \rho^{\otimes 2} (U^\dagger)^{\otimes 2} O^{\otimes 2}]$ . Next we use the same argument as in (a) and obtain the statement. 1pt for the proof. They are expected to say that they use the vectorization identity. But proof steps that intrinsically imply that they used the identity will also be accepted.

- (d) Hence use Eq. (38) to compute  $\int_{U(d)} \text{Tr}[U\rho U^\dagger O]^2 dU$ . What is its value if  $O$  is a Pauli matrix and  $\rho$  is a pure state?  
(4 marks)

Answer:

$$\begin{aligned} & \int_{U(d)} \langle \text{vec}(O^{\otimes 2}) | U^{\otimes 2} \otimes (U^*)^{\otimes 2} | \text{vec}(\rho^{\otimes 2}) \rangle dU \\ &= \frac{1}{d^2 - 1} \left( \langle \text{vec}(O^{\otimes 2}) | \text{vec}(I_{d^2}) \rangle \langle \text{vec}(I_{d^2}) | \text{vec}(\rho^{\otimes 2}) \rangle - \frac{1}{d} \langle \text{vec}(O^{\otimes 2}) | \text{vec}(I_{d^2}) \rangle \langle \text{vec}(F) | \text{vec}(\rho^{\otimes 2}) \rangle \right) \end{aligned} \quad (36)$$

$$- \frac{1}{d} \langle \text{vec}(O^{\otimes 2}) | \text{vec}(F) \rangle \langle \text{vec}(I_{d^2}) | \text{vec}(\rho^{\otimes 2}) \rangle + \langle \text{vec}(O^{\otimes 2}) | \text{vec}(F) \rangle \langle \text{vec}(F) | \text{vec}(\rho^{\otimes 2}) \rangle \quad (37)$$

$$= \frac{1}{d^2 - 1} \left( \text{Tr}[O]^2 \text{Tr}[\rho]^2 - \frac{1}{d} \text{Tr}[O]^2 \text{Tr}[\rho^2] - \frac{1}{d} \text{Tr}[O^2] \text{Tr}[\rho]^2 + \text{Tr}[O^2] \text{Tr}[\rho^2] \right). \quad (38)$$

In case  $O$  is Pauli and  $\rho$  is pure,  $\text{Tr}[O] = 0$  and  $\text{Tr}[O^2]/d = 1 = \text{Tr}[\rho] = \text{Tr}[\rho^2]$ , hence the integral

$$\int_{U(d)} \langle \text{vec}(O^{\otimes 2}) | U^{\otimes 2} \otimes (U^*)^{\otimes 2} | \text{vec}(\rho^{\otimes 2}) \rangle dU = \frac{1}{d^2 - 1} \left( 0 + 0 - \frac{1}{d} \cdot d + d \right) = \frac{1}{d + 1}. \quad (39)$$

2 pts for expression Eq. 28, or its alternative in terms of vectorization inner product. 2 pts for correctly finding the values for  $O$  being Paulis and  $\rho$  pure states. A lot of ppl calculated the trace involving SWAP in a wrong way and hence -1 pt.

Let's now consider a tripartite system consisting of three subspaces of a same dimension  $d$ . We define the quantity

$$X := \text{Tr}[O_{2,3} U_{2,3} (\sigma_2 \otimes \rho_3) U_{2,3}^\dagger] \quad (40)$$

where

$$\sigma_2 = \text{Tr}_1[U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger], \quad (41)$$

$O$  and  $U$ 's have dimension  $d^2$ , the density operators  $\rho$ 's and  $\sigma$  have dimension  $d$ , and we specify the subspace in their subscripts.

(e) Show that

$$X = \text{Tr}[(I_1 \otimes O_{2,3})(I_1 \otimes U_{2,3})(U_{1,2} \otimes I_3)(\rho_1 \otimes \rho_2 \otimes \rho_3)(U_{1,2} \otimes I_3)^\dagger(I_1 \otimes U_{2,3})^\dagger]. \quad (42)$$

(3 marks)

Answer: We use the partial trace identity in cheat sheet - point 19 to pull out the partial trace inside the trace, and obtain the statement. 1 pt for realizing that the operators applying in different subspaces commute. 2 pts for using the identify of pulling out the partial trace.

(f) Use vectorization to show that  $X$  can be written as

$$X = \sum_{Y \in \mathcal{B}} \left( \text{Tr}[(I_1 \otimes Y)U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger] \text{Tr}[O_{2,3}U_{2,3}(Y \otimes \rho_3)U_{2,3}^\dagger] \right) \quad (43)$$

and hence

$$X = \sum_{Y \in \mathcal{B}} \left( \langle \text{vec}(I_1 \otimes Y) | U_{1,2} \otimes U_{1,2}^* | \text{vec}(\rho_1 \otimes \rho_2) \rangle \langle \text{vec}(O_{2,3}) | U_{2,3} \otimes U_{2,3}^* | \text{vec}(Y \otimes \rho_3) \rangle \right) \quad (44)$$

where  $\mathcal{B}$  is a set of operators. Specify a set of operators  $\mathcal{B}$  for which the equality holds. *Hint: you may find it helpful to insert a resolution of the identity.*

(8 marks)

Answer: See QRC paper - appendix Eq. A17-A20. Alternatively one could also use Schmidt decomposition of operators to write  $U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger = \sum_i \alpha_i \tilde{\rho}_1^{(i)} \otimes \tilde{\rho}_2^{(i)}$  and similarly for  $O_{2,3}$ . With this one could easily insert the an identity between their vectorization inner products in each subspaces. No one solved this problem completely. 4 points if they separate the subspaces and insert a copy of space 2. 2-3 pts for trying that. 1 pt if they start with a right direction and did the first steps of moving the terms.

If they show that  $Y$  leads to an identity resolution in the vectorization space and hence  $\mathcal{B}$  must be a basis, they get 4 points. If they guess a choice of  $\mathcal{B}$  correctly and provide some arguments for their choice, they get 2-3 pts.

(g) We let  $U_{1,2} = U_{3,2} = U$ . Under this assumption, use Eq. (38) to compute  $\int_{U(d)} X dU$ .

(4 marks)

2 pt if they find the expression of integral (such that it's ready to apply the Haar integral Eq. 23). -1 pt if they compute two independent first moment integral. 2 pt for Haar integration and computing the sum over  $Y$ .